Phys 410
Fall 2014
Lecture #9 Summary
30 September, 2014

We considered the *driven* damped harmonic oscillator and resonance in detail. The response to a cosine ($\cos(\omega t)$) driving force in the long-term limit is: $x(t) = A\cos(\omega t - \delta)$, where ω is the frequency of the driving force. This represents the long-time persistent solution of the motion. It shows that the oscillator eventually adopts the same frequency as the driving force.

The amplitude function $A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}}$ shows a resonant response. As a function of frequency ω at fixed natural frequency ω_0, there is a maximum amplitude of the persistent motion response when the driving frequency is equal to $\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$. The quality factor of the resonance is a measure of how large and sharply peaked the amplitude response looks. It is defined as the ratio of the frequency at which there is peak energy (or power) amplitude over the frequency bandwidth known as the full-width at half maximum (FWHM). The FWHM is defined as the frequency width at the half-power height. The quality factor, or Q, is given by $Q = \omega_0/2\beta$. As the dissipation (parameterized by β) decreases, the quality factor grows.

The phase evolution through resonance goes from 0 well below resonance to π well above resonance, with $\delta = \pi/2$ exactly at resonance. The slope of $\delta(\omega)$ at resonance is $\frac{1}{\beta} = 2Q/\omega_0$.

We considered several examples of resonant phenomena in mechanical and electrical systems, as noted in the Supplementary Material. One interesting example was that of crowd synchrony on the Millennium bridge in London. The pedestrians on the bridge acted as a set of periodic driving forces on the bridge position. The bridge acted back on the pedestrians in a manner that caused their motion to synchronize and amplify the oscillations of the bridge. This led to closing of the bridge, and modifications to the structure to increase the damping force on the bridge.

The equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$ involves a linear operator $L = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ acting on the displacement function $x(t)$ and relating it to the driving force $f(t)$ as $L\,x(t) = f(t)$. The linearity property means that the operator can operate on any number of solutions at the same time: $L\,[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 f_1(t) + \alpha_2 f_2(t)$, for arbitrary weighting coefficients α_1 and α_2. This property allows us to consider an arbitrary periodic driving force $f(t+T) = f(t)$, where T is the period of the driving force, as being made up of an infinite superposition of cosine driving forces: $f(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t)$, where we assume that a Fourier cosine expansion is adequate to describe the periodic driving force. The linearity of the problem allows us to write down the general solution as $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$, with $A_n = f_n / \sqrt{(\omega_0^2 - (n\omega)^2)^2 + 4\beta^2 n^2 \omega^2}$ and $\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - (n\omega)^2}\right)$. With this we can describe the motion of the driven system subjected to more general driving forces, such as a triangle wave, periodic pulsed driving forces, etc.

We moved on to the question of how to make Newton's Laws of motion work in non-inertial reference frames. This turns out to be useful for a number of reasons. First we often insist on using coordinate systems that are non-inertial, such as the (Latitude, Longitude, Altitude) "GPS" reference frame attached to the surface of the rotating earth. Secondly, some physical problems are easier to attack when seen from non-inertial reference frames, such as the "co-rotating frame" rotating at the Larmor precession frequency in NMR. Another example is the description of small oscillations about an equilibrium point in a non-inertial reference frame.

We considered first the case of a reference frame undergoing constant linear acceleration $\vec{A}$. By comparing a description of the motion of an object as seen from an inertial reference frame to that same object seen from a non-inertial reference frame, we concluded that Newton's second law in the non-inertial reference frame must be written as $m\ddot{\vec{r}} = \vec{F}_{net} - m\vec{A}$. The "inertial force" $\vec{F}_{inertial} = -m\vec{A}$ must be added to the net force to make the equation of motion work in the non-inertial frame. We experience this inertial force as a backwards force when sitting in an aircraft that is accelerating down the runway for takeoff.

Making Newton's second law work in a rotating reference frame is more of a challenge. Consider a rigid body moving through space. A rigid body is one in which the distances between the particles do not change during the motion. We can start by describing the motion of the center of mass $\vec{R}_{CM}(t)$ and treat it as the motion of a particle of mass M equal to the total mass of the object. With an extended rigid body we have the additional degree of freedom that the object can also be rotating or tumbling. We can treat the center of mass as a stationary point during the motion. Euler's theorem says that the most general motion of that object is a rotation about an axis going through the center of mass. This rotational motion is specified by a direction of the rotation axis and the magnitude of the rotation rate. Rotation is specified by an axis of rotation $\hat{u}$, and a rate ω, so that $\vec{\omega} = \omega\hat{u}$. The rotation axis goes through the fixed point in the object. We found that the linear velocity of a particle at location $\vec{r}$ inside or on the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$.